

# Note on the oriented diameter of graphs with diameter 3\*

Hengzhe Li, Xueliang Li, Yuefang Sun, Jun Yue

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, China

lh2010@mail.nankai.edu.cn; lxl@nankai.edu.cn

bruceun@gmail.com; yuejun06@126.com

## Abstract

In 1978, Chvátal and Thomassen showed that every bridgeless graph with diameter 2 has an orientation with diameter at most 6. They also gave general bounds on the smallest value  $f(d)$  such that every bridgeless graph  $G$  with diameter  $d$  has an orientation with diameter at most  $f(d)$ . For  $d = 3$ , they proved that  $8 \leq f(d) \leq 24$ . Until recently, Kwok, Liu and West improved the above bounds by proving  $9 \leq f(3) \leq 11$  in [P.K. Kwok, Q. Liu and D.B. West, Oriented diameter of graphs with diameter 3, J. Combin. Theory Ser.B 100(2010), 265-274]. In this paper, we determine the oriented diameter among the bridgeless graphs with diameter 3 that have minimum number of edges.

**Keywords:** diameter, oriented diameter, extremal graph.

**AMS subject classification 2010:** 05C12, 05C20.

---

\*Supported by NSFC No.11071130.

# 1 Introduction

All graphs in this paper are finite and simple. A *graph*  $G$  is an ordered pair  $(V(G), E(G))$  consisting of a set  $V(G)$  of *vertices* and a set  $E(G)$  of *edges*. We refer to book [1] for graph theoretical notation and terminology not given here. An orientation of a graph  $G$  is a *digraph* obtained from  $G$  by replacing each edge by just one of the two possible arcs with the same ends. We occasionally use the symbol  $\vec{G}$  to specify an orientation of  $G$  (even though a graph generally has many orientations). An orientation of a simple graph is referred to as an *oriented graph*. The diameter of a graph  $G$  (digraph  $D$ ) is  $\max\{d(u, v) \mid u, v \in V(G)\}$  ( $\max\{d(u, v) \mid u, v \in V(D)\}$ ), denoted by  $\text{diam}(G)$  ( $\text{diam}(D)$ ). The oriented diameter of a bridgeless graph is  $\min\{\text{diam}(\vec{G}) \mid \vec{G} \text{ is an orientation of } G\}$ .

In 1939, Robbins solved the One-Way Street Problem and proved that a graph  $G$  admits a strongly connected orientation if and only if  $G$  is bridgeless, that is, does not have any cut-edge. Naturally, one hopes that the oriented diameter of a bridgeless graph is as small as possible. Bondy and Murty suggested to study the quantitative variations on Robbins' theorem. In particular, they conjectured that there exists a function  $f$  such that every bridgeless graph with diameter  $d$  admits an orientation of diameter at most  $f(d)$ .

In 1978, Chvátal and Thomassen [7] gave general bounds  $\frac{1}{2}d^2 + d \leq f(d) \leq 2d^2 + 2d$  for  $d \geq 2$ . These bounds have not got improved in the past several decades. They also proved that  $f(1) = 3$  and  $f(2) = 6$ . For  $d = 3$ , the general result reduces to  $8 \leq f(3) \leq 24$ . Until recently, Kwok, Liu and West improved the above bounds by proving  $9 \leq f(3) \leq 11$  in [12].

In [7, 12], they got the bounds by considering the distances from two special adjacent vertices to the other vertices, the distances from the other vertices to the two special vertices, and the triangle inequality of distances. In this paper, we will directly consider the distance from one vertex to another in bridgeless graphs with diameter 3 that have minimum number of edges.

Chvátal and Thomassen [7] showed that determining whether an arbitrary graph may be oriented so that its diameter is at most 2 is NP-complete.

Bounds of oriented diameter of graphs have also been studied in terms of other parameters, for example, radius, dominating number [7, 8, 16], etc. Some classes of graphs have also been studied in [8, 9, 10, 11, 13].

This paper is organized as follows: in Section 2, we give some known results and introduce a class of extremal bridgeless graphs with diameter 3; in Section 3, we present a vertex set partition of this graph class, and specify an orientation under this partition; in Section 4, we show the above orientation is optimal.

## 2 Preliminaries

In [12], Kwok, Liu and West gave the following proposition by constructing the graph of Figure 1a.

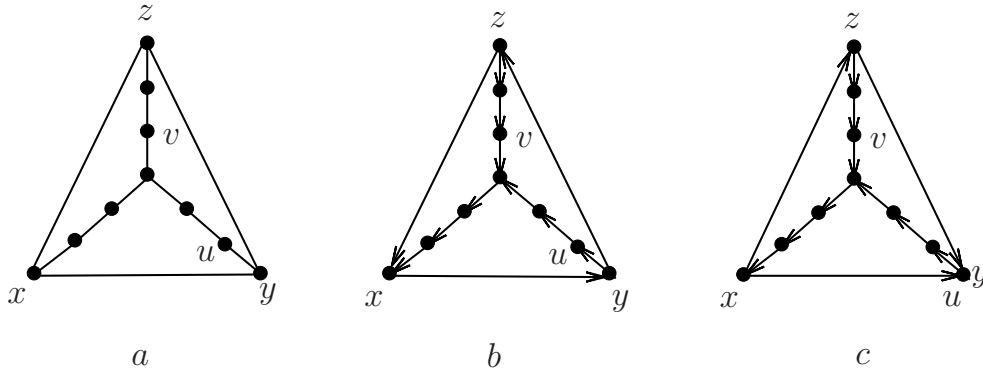


Figure 1. A graph of diameter 3 and its two oriented graphs with diameter 9.

**Proposition 1.** [12]

$$f(3) \geq 9.$$

Clearly, the distance from  $u$  to  $v$  is 9 in Figures 1b and 1c. (each oriented graph is isomorphic to one of Figures 1a and 1b).

Let  $S$  and  $S'$  be two disjoint vertex sets. We use  $E[S, S']$  to denote the set of edges having one end in each one of  $S$  and  $S'$ .

**Lemma 1.** [12] *In a graph  $H$ , let  $S$  and  $S'$  be disjoint vertex sets such that  $S' \subseteq N_H(S)$ , and let  $F$  denote the graph  $H[S'] \cup (S \cup S', E[S', S])$ . If the induced subgraph  $H[S']$  is connected and nontrivial, then there is an orientation  $\vec{F}$  of  $F$  such that  $d_{\vec{F}}(S, w) \leq 2$  and  $d_{\vec{F}}(w, S) \leq 2$  for every  $w \in S'$ .*

**Remark 1.** *In fact, the condition “the induced subgraph  $H[S']$  is connected and nontrivial” can be replaced by “the induced subgraph  $H[S']$  does not have trivial components” by the proof of Lemma 1.*

Now, we introduce a class of extremal graph. Denote by  $G(n, k, \lambda, s)$  the class of graphs with  $n$  vertices and diameter at most  $k$  which have the property that by deleting any  $s$  or fewer edges the resulting subgraphs have diameters at most  $\lambda > k$ . Furthermore, denote by  $\text{Min}G(n, k, \lambda, s)$  the subclass (of  $G(n, k, \lambda, s)$ ) of graphs with minimum number of edges, and denote by  $M(n, k, \lambda, s)$  the minimum possible number of edges.

In [6], Caccetta gave the following observation and lemma.

**Observation 1.** [6]

(1) *A  $G(n, k, \lambda, s)$  graph is also a  $G(n, k', \lambda', s')$  graph, whenever  $k' \geq k$ ,  $\lambda' \geq \lambda$  and  $s' \leq s$ . Consequently, the function  $M(n, k, \lambda, s)$  is monotonic non-decreasing in  $s$ , and monotonic non-increasing in  $k$  and  $\lambda$ .*

(2) *In a  $G(n, k, \lambda, s)$  there will be at least  $s + 1$  edge disjoint paths of length  $\leq \lambda$  between any two vertices, at least one of which has length  $\leq k$ .*

(3) *The degree of every vertex of  $G$  is at least  $s + 1$ , that is,  $\delta(G) \geq s + 1$ .*

(4) *If  $\delta(G) = s + 1$ , then every vertex of  $G$  which is not adjacent to  $x$  with degree  $\delta(G)$  must be connected to each of the  $s + 1$  vertices adjacent to  $x$  by a path of length  $\leq \lambda - 1$  (from (2)).*

**Lemma 2.** [6] *Let  $G \in \text{Min}G(n, 3, \lambda, 1)$ , where  $\lambda \geq 4$ . Then  $G$  possesses two adjacent vertices of degree 2 for every  $n \geq 5$  except possibly  $n = 8, 9, 10$  and 12. Furthermore, if  $G$  does not possess two adjacent vertices of degree 2, then the only possible structures are the graphs  $H^8, H^9, H_j^{10} (j = 1, 2, 3, 4)$  and  $H_j^{12} (j = 1, 2, 3)$  in Figure 2.*

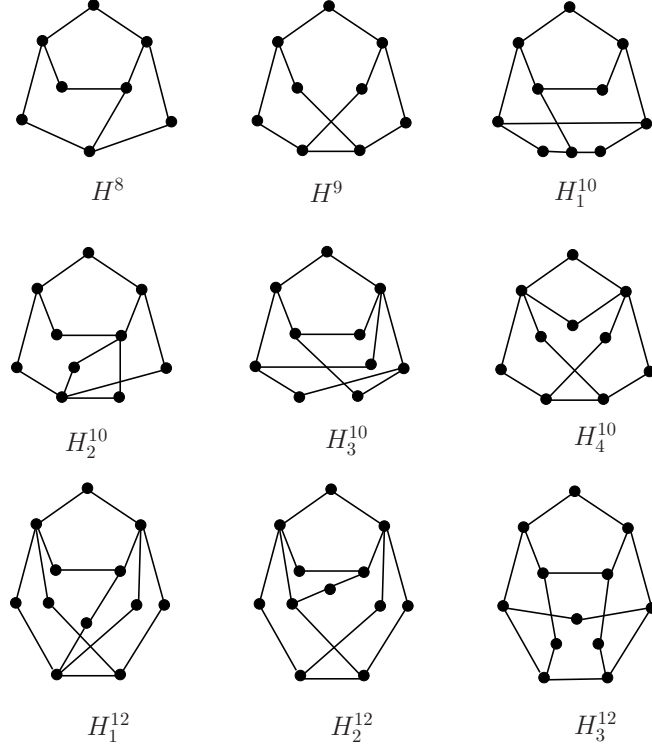


Figure 2. The graphs  $H^8, H^9, H_j^{10} (j = 1, 2, 3, 4)$  and  $H_j^{12} (j = 1, 2, 3)$ .

There are many interesting results on  $\text{Min}G(n, k, \lambda, s)$ . We refer the readers to [2, 3, 4, 5, 6] for some more results or details.

### 3 Vertex set partition and orientation

In this section, let  $G \in \text{Min}G(n, 3, \lambda, 1)$ , where  $n \geq 5$  and  $\lambda \geq 4$ , and let  $u$  and  $v$  be two adjacent vertices of degree 2 in  $G$ . Suppose  $u(v)$  is adjacent to  $x(y)$ . Let  $X, Y$  and  $Z$  denote the sets  $N(x) \setminus N(y)$ ,  $N(y) \setminus N(x)$  and  $N(x) \cap N(y)$ , respectively.

Let  $A = X \cup Y \cup Z \cup \{u, v, x, y\}$ . For  $s \in V(G) \setminus A$ , clearly  $d_G(s, u) = d_G(s, v) = 3$  since  $G \in G(n, 3, \lambda, 1)$ , that is  $N(s) \cap N(x) \neq \emptyset$  and  $N(s) \cap N(y) \neq \emptyset$ . We partition this set based on the distribution of the neighbors

of  $s$ .

$$\begin{aligned} W &= (N(X) \cap N(Y)) \setminus A; \\ I &= (N(X) \cap N(Z)) \setminus A; \\ K &= (N(Y) \cap N(Z)) \setminus A; \\ J &= V(G) \setminus (W \cup I \cup K \cup A). \end{aligned}$$

See Figure 3 for details.

At this point, we further partition  $X$  and  $Y$  as follows:

$$\begin{aligned} X_1 &= \{x \in X \mid x \text{ has neighbors in } Y \cup Z \cup I \cup W\}, \\ X_2 &= \{x \in X \setminus X_1 \mid x \text{ is an isolated vertex in } G[X \setminus X_1]\}, \\ X_3 &= X \setminus (X_1 \cup X_2), \\ Y_1 &= \{y \in Y \mid y \text{ has neighbors in } X \cup Z \cup K \cup W\}, \\ Y_2 &= \{y \in Y \setminus Y_1 \mid y \text{ is an isolated vertex in } G[Y \setminus Y_1]\}, \\ Y_3 &= Y \setminus (Y_1 \cup Y_2). \end{aligned}$$

Note that, in Figure 3, if  $s$  and  $t$  lie in distinct ellipses and there exists no edge joining the two ellipses, then  $s$  and  $t$  are nonadjacent in  $G$ . Generally, the edges drawn in Figure 3 do not indicate complete bipartite subgraph.

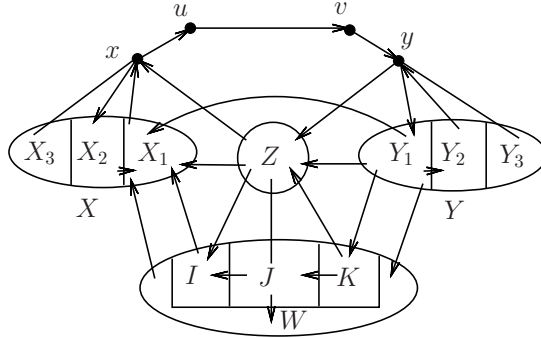


Figure 3. An optimal orientation of  $G$ .

Given this partition, now we can give an orientation of some edges of  $G$ , as shown in Figure 3. For the vertex sets (or vertices)  $S$  and  $T$ , we use the notation  $S \rightarrow T$  to mean that all edges with endpoints in  $S$  and  $T$  are

oriented from  $S$  to  $T$ . Thus for every sequence below, all edges with ends in two successive sets are oriented from the first set to the second.

$$\begin{aligned} u &\rightarrow v \rightarrow y \rightarrow Y_1 \rightarrow W \rightarrow X_1 \rightarrow x \rightarrow u, \\ y &\rightarrow Z \rightarrow x, \quad Y_1 \rightarrow Z \rightarrow X_1, \\ Y_1 &\rightarrow K \rightarrow Z, \quad Z \rightarrow I \rightarrow X_1, \\ K &\rightarrow J \rightarrow I, \quad J \rightarrow W, \end{aligned}$$

$$x \rightarrow X_2 \rightarrow X_1, \quad Y_1 \rightarrow Y_2 \rightarrow y.$$

We further partition  $J$  as follows:

$$\begin{aligned} J_1 &= J \cap N(K), \quad J_2 = (J \cap N(I)) \setminus J_1, \quad J_3 = (J \cap N(W)) \setminus (J_1 \cup J_2), \\ J_4 &= J \setminus (J_1 \cup J_2 \cup J_3) = \{s \in J \mid s \text{ has no neighbor in } I \cup K \cup W\}, \\ J_{4,1} &= \{s \in J_4 \mid N(s) \in Z\}, \\ J_{4,2} &= J_4 \setminus J_{4,1} = \{s \in J_4 \mid s \in N(Z) \cup N(J)\}. \end{aligned}$$

We further orient the edges of  $G$  as follows:

$$J_1 \rightarrow Z, \quad Z \rightarrow J_2, \quad Z \rightarrow J_3.$$

Since  $G \in \text{Min}G(n, 3, \lambda, 1)$ , then  $\delta(G) \geq 2$  by Observation 1. Thus, for any  $s \in J_{4,1}$ , we oriented one of  $E[s, Z]$  from  $s$  to  $Z$ , and the others from  $Z$  to  $s$ . By the above definition, we know that  $G[J_{4,2}]$  does not have trivial components.

Clearly, by Lemma 1 and Remark 1, there exist orientations  $F_1, F_2$  and  $F_3$  of graphs  $G[J_{4,2}] \cup (J_{4,2} \cup Z, E[J_{4,2}, Z])$ ,  $G[X_3] \cup (X_3 \cup \{x\}, E[X_3, \{x\}])$  and  $(G[Y_3] \cup (Y_3 \cup \{y\}, E[Y_3, \{y\}]))$ , respectively, such that for any  $s_1 \in J_{4,2}$ ,  $s_2 \in X_3$  and  $s_3 \in Y_3$ , we have  $d_{F_1}(s_1, Z) \leq 2$ ,  $d_{F_1}(Z, s_1) \leq 2$ ,  $d_{F_2}(s_2, x) \leq 2$ ,  $d_{F_2}(x, s_2) \leq 2$ ,  $d_{F_3}(s_3, y) \leq 2$  and  $d_{F_3}(y, s_3) \leq 2$ . The other edges can be oriented arbitrarily.

## 4 Determine oriented diameter

In this section, we show that every  $G \in \text{Min}G(n, 3, \lambda, 1)$  has an oriented diameter at most 9.

For the orientation in Section 3, we have the following three observations.

**Observation 2.** (1) For every  $x \in X_2$ ,  $N(x) \cap X_1 \neq \emptyset$ ; (2) For every  $y \in Y_2$ ,  $N(y) \cap Y_1 \neq \emptyset$ .

It is easy to see that this observation holds by the definition of the vertex set partition.

**Observation 3.** (1) For every  $s \in X_1$ , there exists a path with length at most 3 from  $y$  to  $s$  in the oriented graph  $\vec{G}$ ; (2) For every  $s \in Y_1$ , there exists a path with length at most 3 from  $s$  to  $x$  in the oriented graph  $\vec{G}$ .

*Proof.* Since the proof methods are similar, we only show (1). For any  $s \in X_1$ , by the definition of set  $X_1$ , we know that  $s$  has neighbors in  $Y \cup Z \cup I \cup W$ . Let  $t$  be a neighbor of  $s$  in  $Y \cup Z \cup I \cup W$ . If  $t \in Y$ , then  $t \in Y_1$  by the definition of set  $Y_1$ . Thus  $y, t, s$  is a path with length 2 from  $y$  to  $s$ . If  $t \in Z$ , then  $y, t, s$  is a path with length 2 from  $y$  to  $s$ . If  $t \in I$ , then  $y, z, t, s$  is a path with length 3 from  $y$  to  $s$ , where  $z$  is a neighbor of  $t$  in  $Z$  (Note that such a vertex  $z$  must exist by the definition of set  $I$ . Otherwise  $t \in W$ , then  $y, y', t, s$  is a path with length 3 from  $y$  to  $s$ , where  $y'$  is a neighbor of  $t$  in  $Y_1$ ).  $\square$

**Observation 4.** If  $Z \neq \emptyset$ , then (1) for every  $s \in X_2 \cup X_3$ , there exists a path with length at most 4 from  $y$  to  $s$  in the oriented graph  $\vec{G}$ ; (2) for every  $s \in Y_2 \cup Y_3$ , there exists a path with length at most 4 from  $s$  to  $x$  in the oriented graph  $\vec{G}$ .

*Proof.* Since the proof methods are similar, we only show (1). Since  $Z \neq \emptyset$ , we pick any  $z \in Z$ . Then  $y, z, x$  is a path with length 2 from  $y$  to  $x$ . If  $s \in X_2$ , then  $y, z, x, s$  is a path with length 3 from  $y$  to  $s$ . Otherwise, that is,  $s \in X_3$ , there exists a path  $P$  with length at most 2 from  $x$  to  $s$  by Lemma 1 and Remark 1. Thus, a path with length at most 4 from  $y$  to  $s$  is obtained from the path  $y, z, x$  and  $P$ .  $\square$

**Lemma 3.** Let  $G \in \text{MinG}(n, 3, \lambda, 1)$ , where  $n \geq 5$  and  $\lambda \geq 4$ . If  $G$  possesses two adjacent vertices of degree 2, then the oriented diameter of  $G$  is at most 9.



*Proof.* We show that the oriented graph  $\vec{G}$  in Section 3 has a diameter at most 9. Consider the following two cases.

**Case 1.**  $Z \neq \emptyset$ .

We only show that for any  $(s, t) \in (J, V(G) \setminus J), (V(G) \setminus J, J)$  and  $(J, J)$ , the distance from  $s$  to  $t$  is at most 9, since it is easy to check the remaining cases hold.

	$J_1$	$J_2$	$J_3$	$J_{4,1}$	$J_{4,2}$
$x$	$x, u, v, y, Y_1, K, J_1$	$x, u, v, y, Z, J_2$	$x, u, v, y, Z, J_3$	$x, u, v, y, Z, J_{4,1}$	$x, u, v, y, Z$ and Lemma 1
$u$	$u, v, y, Y_1, K, J_1$	$u, v, y, Z, J_2$	$u, v, y, Z, J_3$	$u, v, y, Z, J_{4,1}$	$u, v, y, Z$ and Lemma 1
$v$	$v, y, Y_1, K, J_1$	$v, y, Z, J_2$	$v, y, Z, J_3$	$v, y, Z, J_{4,1}$	$v, y, Z$ and Lemma 1
$y$	$y, Y_1, K, J_1$	$y, Z, J_2$	$y, Z, J_3$	$y, Z, J_{4,1}$	$y, Z$ and Lemma 1
$Y_1$	Obs 3 and $x, u, v, y, Y_1, K, J_1$	Obs 3 and $x, u, v, y, Z, J_2$	Obs 3 and $x, u, v, y, Z, J_3$	Obs 3 and $x, u, v, y, Z, J_{4,1}$	Obs 3, $x, u, v, y, Z$ and Lemma 1
$Y_2$	$Y_2, y, Y_1, K, J_1$	$Y_2, y, Z, J_2$	$Y_2, y, Z, J_3$	$Y_2, y, Z, J_{4,1}$	$Y_2, y, Z$ and Lemma 1
$Y_3$	Lemma 1 and $y, Y_1, K, J_1$	Lemma 1 and $y, Z, J_2$	Lemma 1 and $y, Z, J_3$	Lemma 1 and $y, Z, J_{4,1}$	Lemma 1, $y, Z$ and Lemma 1
$K$	$K, Z, x, u, v, y, Y_1, K, J_1$	$K, Z, x, u, v, y, Z, J_2$	$K, Z, x, u, v, y, Z, J_3$	$K, Z, x, u, v, y, Z, J_{4,1}$	$K, Z, x, u, v, y, Z$ and Lemma 1
$W$	$W, X_1, x, u, v, y, Y_1, K, J_1$	$W, X_1, x, u, v, y, Z, J_2$	$W, X_1, x, u, v, y, Z, J_3$	$W, X_1, x, u, v, y, Z, J_{4,1}$	$W, X_1, x, u, v, y, Z$ and Lemma 1
$Z$	$Z, x, u, v, y, Y_1, K, J_1$	$Z, x, u, v, y, Z, J_2$	$Z, x, u, v, y, Z, J_3$	$Z, x, u, v, y, Z, J_{4,1}$	$Z, x, u, v, y, Z$ and Lemma 1
$I$	$I, X_1, x, u, v, y, Y_1, K, J_1$	$I, X_1, x, u, v, y, Z, J_2$	$I, X_1, x, u, v, y, Z, J_3$	$I, X_1, x, u, v, y, Z, J_{4,1}$	$I, X_1, x, u, v, y, Z$ and Lemma 1
$X_1$	$X_1, x, u, v, y, Y_1, K, J_1$	$X_1, x, u, v, y, Z, J_2$	$X_1, x, u, v, y, Z, J_3$	$X_1, x, u, v, y, Z, J_{4,1}$	$X_1, x, u, v, y, Z$ and Lemma 1
$X_2$	$X_2, X_1, x, u, v, y, Y_1, K, J_1$	$X_2, X_1, x, u, v, y, Z, J_2$	$X_2, X_1, x, u, v, y, Z, J_3$	$X_2, X_1, x, u, v, y, Z, J_{4,1}$	$X_2, X_1, x, u, v, y, Z$ and Lemma 1
$X_3$	Lemma 1 and $x, u, v, y, Y_1, K, J_1$	Lemma 1 and $x, u, v, y, Z, J_2$	Lemma 1 and $x, u, v, y, Z, J_3$	Lemma 1 and $x, u, v, y, Z, J_{4,1}$	Lemma 1, $x, u, v, y, Z$ and Lemma 1

Table 1. Directed paths in  $\vec{G}$ .

If  $(s, t) \in (V(G) \setminus J, J)$ , then a path with length at most 9 from  $s$  to  $t$  is presented in Table 1. The notation  $X_1, X_2, \dots, X_k$  means that there exists some desired path  $P = (x_1, x_2, \dots, x_k)$ , where  $x_i \in X_i$ ,  $i = 1, 2, \dots, k$ .

Note that, in Figure 3, at least two of  $X_1, Y_1, Z$  are nonempty; some sets may be empty (if so, the proof becomes easier). Though we have not indicated all kinds of cases, in fact, we can get information from these sequences

	$J_1$	$J_2$	$J_3$	$J_{4,1}$	$J_{4,2}$
$J_1$	$J_1, Z, x, u, v, y, Y_1, K, J_1$	$J_1, Z, x, u, v, y, Z, J_2$	$J_1, Z, x, u, v, y, Z, J_3$	$J_1, Z, x, u, v, y, Z, J_{4,1}$	$J_1, Z, x, u, v, y, Z$ and Lemma 1
$J_2$	$J_2, I, X_1, x, u, v, y, Y, K, J_1$	$J_2, I, X_1, x, u, v, y, Z, J_2$	$J_2, I, X_1, x, u, v, y, Z, J_3$	$J_2, I, X_1, x, u, v, y, Z, J_{4,1}$	$J_2, I, X_1, x, u, v, y, Z$ and Lemma 1
$J_3$	$J_3, W, X_1, x, u, v, y, Y, K, J_1$	$J_3, W, X_1, x, u, v, y, Z, J_2$	$J_3, W, X_1, x, u, v, y, Z, J_3$	$J_3, W, X_1, x, u, v, y, Z, J_{4,1}$	$J_3, W, X_1, x, u, v, y, Z$ and Lemma 1
$J_{4,1}$	$J_{4,1}, Z, x, u, v, y, Y, K, J_1$	$J_{4,1}, Z, x, u, v, y, Z, J_2$	$J_{4,1}, Z, x, u, v, y, Z, J_3$	$J_{4,1}, Z, x, u, v, y, Z, J_{4,1}$	$J_{4,1}, Z, x, u, v, y, Z$ and Lemma 1
$J_{4,2}$	Lemma 1 and $Z, x, u, v, y, Y_1, K, J_1$	Lemma 1 and $Z, x, u, v, y, Z, J_2$	Lemma 1 and $Z, x, u, v, y, Z, J_3$	Lemma 1 and $Z, x, u, v, y, Z, J_{4,1}$	Lemma 1 and $Z, x, u, v, y, Z$ and Lemma 1

Table 2. Directed paths in  $\vec{G}$

of Table 1. For example, if  $(s, t) \in (X_3, J_3)$ , then  $X_3$  and  $J_3$  are nonempty. By the definition of set  $J_3$ , we know that  $W, X_1$  and  $Y_1$  are also nonempty. In Table 1,  $(s, t) \in (X_3, J_3)$  corresponds to “Lemma 1 and  $x, u, v, y, Z, J_3$ ”. “Lemma 1” means that there exists a path with length at most 2 from  $s$  to  $x$  by the definition of the above orientation and Lemma 1.

If  $(s, t) \in (J, J)$ , then a path with length at most 9 from  $s$  to  $t$  is presented in Table 2. If  $(s, t) \in (J, V(G) \setminus J)$ , then a path with length at most 9 from  $s$  to  $t$  is presented in Table 3.

	$x$	$u$	$v$	$y$	$Y_1$	$Y_2$	$Y_3$
$J_1$	$J_1, Z, x$	$J_1, Z, x, u$	$J_1, Z, x, u, v$	$J_1, Z, x, u, v, y$	$J_1, Z, x, u, v, y, Y_1$	$J_1, Z, x, u, v, y, Y_1, Y_2$	$J_1, Z, x, u, v, y$ and Lemma 1
$J_2$	$J_2, I, X_1, x$	$J_2, I, X_1, x, u$	$J_2, I, X_1, x, u, v$	$J_2, I, X_1, x, u, v, y$	$J_2, I, X_1, x, u, v, y, Y_1$	$J_2, I, X_1, x, u, v, y, Y_1, Y_2$	$J_2, I, X_1, x, u, v, y$ and Lemma 1
$J_3$	$J_3, W, X_1, x$	$J_3, W, X_1, x, u$	$J_3, W, X_1, x, u, v$	$J_3, W, X_1, x, u, v, y$	$J_3, W, X_1, x, u, v, y, Y_1$	$J_3, W, X_1, x, u, v, y, Y_1, Y_2$	$J_3, W, X_1, x, u, v, y$ and Lemma 1
$J_{4,1}$	$J_{4,1}, Z, x$	$J_{4,1}, Z, x, u$	$J_{4,1}, Z, x, u, v$	$J_{4,1}, Z, x, u, v, y$	$J_{4,1}, Z, x, u, v, y, Y_1$	$J_{4,1}, Z, x, u, v, y, Y_1, Y_2$	$J_{4,1}, Z, x, u, v, y$ and Lemma 1
$J_{4,2}$	Lemma 1 and $Z, x$	Lemma 1 and $Z, x, u$	Lemma 1 and $Z, x, u, v$	Lemma 1 and $Z, x, u, v, y$	Lemma 1 and $Z, x, u, v, y, Y_1$	Lemma 1 and $Z, x, u, v, y, Y_1, Y_2$	Lemma 1 and $Z, x, u, v, y$ and Lemma 1

Table 3. Directed paths in  $\vec{G}$ .

**Case 2.**  $Z = \emptyset$ . By definition of the above set partition, we know that  $I = J = K = \emptyset$ , and this case can be checked easily. Observation 4 is used in this case.  $\square$

	$K$	$W$	$Z$	$I$	$X_1$	$X_2$	$X_3$
$J_1$	$J_1, Z, x,$ $u, v, y,$ $Y, K$	$J_1, Z, x,$ $u, v, y,$ $Y, W$	$J_1, Z, x,$ $u, v, y,$ $Z$	$J_1, Z, x,$ $u, v, y,$ $Z, I$	$J_1, Z, x, u, v, y,$ and Obs 3	$J_1, Z, x, X_2$	$J_1, Z, x,$ and Lemma 1
$J_2$	$J_2, I, X_1,$ $x, u, v, y,$ $Y_1, K$	$J_2, I, X_1,$ $x, u, v, y,$ $Y_1, W$	$J_2, I, X_1,$ $x, u, v, y,$ $Z$	$J_2, I, X,$ $x, u, v, y,$ $Z, I$	$J_2, I, X_1, x, u, v,$ $y,$ and Obs 3	$J_2, I, X_1, x,$ $X_2$	$J_2, I, X_1, x$ and Lemma 1
$J_3$	$J_3, W, X,$ $x, u, v,$ $y, Y, K$	$J_3, W, X,$ $x, u, v,$ $y, Y, W$	$J_3, W, X,$ $x, u, v,$ $y, Z$	$J_3, W, X,$ $x, u, v,$ $y, Z, I$	$J_3, W, X,$ $x, u, v,$ $y,$ and Obs 3	$J_3, W, X, x,$ $X_2$	$J_3, W, X, x,$ and Lemma 1
$J_{4,1}$	$J_{4,1}, Z, x,$ $u, v, y,$ $Y_1, K$	$J_{4,1}, Z, x,$ $u, v, y,$ $Y_1, W$	$J_{4,1}, Z, x,$ $u, v, y,$ $Z$	$J_{4,1}, Z, x,$ $u, v, y,$ $Z, I$	$J_{4,1}, Z, x, u, v, y,$ and Obs 3	$J_{4,1}, Z, x,$ $X_2$	$J_{4,1}, Z, x,$ and Lemma 1
$J_{4,2}$	Lemma 1, $Z, x, u,$ $v, y, Y,$ $K$	Lemma 1 $Z, x, u,$ $v, y, Y,$ $W$	Lemma 1 $Z, x, u,$ $v, y, Z$	Lemma 1 $Z, x, u,$ $v, y, Z, I$	Lemma 1, $Z,$ $x, u, v, y,$ and Obs 3	Lemma 1, $Z, x, X_2$	Lemma 1 $Z,$ $x,$ and Lemma 1

Table 4. Directed paths in  $\vec{G}$ .

**Lemma 4.** *Let  $G \in \text{MinG}(n, 3, \lambda, 1)$ , where  $n \geq 5$  and  $\lambda \geq 4$ . If  $G$  does not possess two adjacent vertices of degree 2, then the oriented diameter of  $G$  is at most 9.*

*Proof.* Let  $G \in \text{MinG}(n, 3, \lambda, 1)$  which does not possess two adjacent vertices of degree 2, where  $n \geq 5$  and  $\lambda \geq 4$ . By Lemma 2,  $G$  must be one of the graphs  $H^8, H^9, H_j^{10} (j = 1, 2, 3, 4)$  and  $H_j^{12} (j = 1, 2, 3)$  in Figure 3. It suffices to show that the oriented diameters of these graphs is at most 9. In fact, we can orient these graphs as Figure 4. It is easy to check that the diameters of  $\vec{H}^8, \vec{H}^9, \vec{H}_j^{10} (j = 1, 2, 3, 4)$  and  $\vec{H}_j^{12} (j = 1, 2, 3)$  are at most 9. Thus we are done.  $\square$

Combining Lemmas 3 and 4, we know that the following theorem holds.

**Theorem 1.** *Let  $G \in \text{MinG}(n, 3, \lambda, 1)$ , where  $n \geq 5$  and  $\lambda \geq 4$ . Then the oriented diameter of  $G$  is at most 9.*

Since the oriented diameter of  $G$  is no larger than the oriented diameter of its spanning subgraph. We have the following corollary.

**Corollary 1.** *If a graph  $G$  has a spanning bridgeless subgraph with diameter at most 3, which admits two adjacent vertices of degree 2, then the oriented*

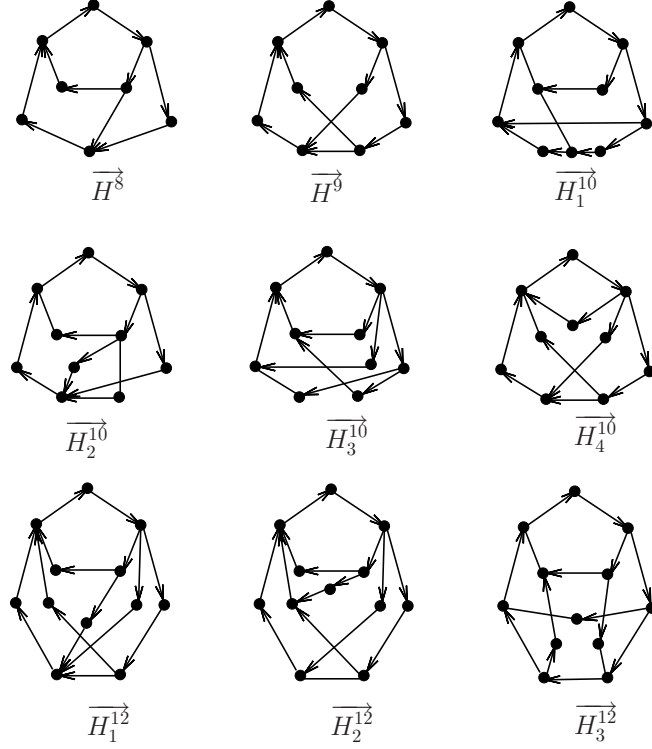


Figure 4. the orientations of  $H^8, H^9, H_j^{10}(j = 1, 2, 3, 4)$  and  $H_j^{12}(j = 1, 2, 3)$ .

diameter of  $G$  is at most 9.

**Corollary 2.** *If a graph  $G$  admits a spanning subgraph contained in  $\text{Min}G(n, 3, \lambda, 1)$ , then the oriented diameter of  $G$  is at most 9.*

By the above arguments, we propose the following conjecture.

**Conjecture 1.** *Every bridgeless graphs with diameter at most 3 has an oriented diameter at most 9.*

If the above conjecture is true, then by Proposition 1, one can get that  $f(3) = 9$ .

## References

- [1] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, 2008.

- [2] B. Bollobas, Graphs of given diameter, in: Proceedings of Colloquium held at Tihany 1968 (ed. P. Erdos and G. Katona), pp.29-36.
- [3] B. Bollobas, Graphs with given diameter and minimal degree, *Ars Combin.* 2(1976), 3-9.
- [4] B. Bollobas, F. Harary, Extremal graphs with given diameter and connectivity, *Ars Combin.* 1(1976), 281-296.
- [5] L. Caccetta, A problem in extremal graph theory, *Ars Combin.* 2(1976), 33-56.
- [6] L. Caccetta, Extremal graphs of diameter 3, *J. Austral. Math. Soc. (Ser. A)* 28(1979), 67-81.
- [7] V. Chvátal, C. Thomassen, Distances in orientations of graphs, *J. Combin. Theory, Ser.B*, 24(1978), 61-75.
- [8] F.V. Fomin, M. Matamala, E. Prisner, I. Rapaport, AT-free graphs: Linear bounds for the oriented diameter, *Discrete Appl. Math.* 141(2004), 135-148.
- [9] K.M. Koh, B.P. Tan, The minimum diameter of orientations of complete multipartite graphs, *Graphs & Combin.* 12(1996), 333-339.
- [10] K.M. Koh, E.G. Tay, Optimal orientations of products of paths and cycles, *Discrete Appl. Math.* 78(1997), 163-174.
- [11] J.C. König, D.W. Krumme, E. Lazard, Diameter-preserving orientations of the torus, *Networks* 32(1998), 1-11.
- [12] P.K. Kwok, Q. Liu, D.B. West, Oriented diameter of graphs with diameter 3, *J. Combin. Theory, Ser.B*, 100(2010), 265-274.
- [13] J.E. Mccanna, Orientations of the  $n$ -cube with minimum diameter, *Discrete Math.* 68(1988), 309-310.
- [14] J. Plesník, Remarks on diameters of orientations of graphs, *Acta Math. Univ. Comenian.* 46/47(1985), 225-236.

- [15] H. Robbins, A theorem on graphs with an application to a problem of traffic control, Amer. Math. Monthly 46(1939), 281-283.
- [16] L. Šoltés, Orientations of graphs minimizing the radius or the diameter, Math. Slovaca 36(1986), 289-296.